

# DERIVED CATEGORIES OF COHERENT SHEAVES AND MOTIVES OF K3 SURFACES

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**ABSTRACT.** Let  $X$  and  $Y$  be smooth complex projective varieties. We will denote by  $D^b(X)$  and  $D^b(Y)$  their derived categories of bounded complexes of coherent sheaves;  $X$  and  $Y$  are derived equivalent if there is a  $\mathbb{C}$ -linear equivalence  $F: D^b(X) \xrightarrow{\sim} D^b(Y)$ . Orlov conjectured that if  $X$  and  $Y$  are derived equivalent then their motives  $M(X)$  and  $M(Y)$  are isomorphic in Voevodsky's triangulated category of motives  $DM_{gm}(\mathbb{C})$  with  $\mathbb{Q}$ -coefficients. In this paper we prove the conjecture in the case  $X$  is a K3 surface admitting an elliptic fibration (a case that always occurs if the Picard rank  $\rho(X)$  is at least 5) with finite-dimensional Chow motive. We also relate this result with a conjecture by Huybrechts showing that, for a K3 surface with a symplectic involution  $f$ , the finite-dimensionality of its motive implies that  $f$  acts as the identity on the Chow group of 0-cycles. We give examples of pairs of K3 surfaces with the same finite-dimensional motive but not derived equivalent.

## 1. INTRODUCTION

Let  $X$  be a smooth projective variety over  $\mathbb{C}$ . We will denote by  $D^b(X)$  the derived category of bounded complexes of coherent sheaves on  $X$ . We say that two smooth projective varieties  $X$  and  $Y$  are **derived equivalent** if there is a  $\mathbb{C}$ -linear equivalence  $F: D^b(X) \xrightarrow{\sim} D^b(Y)$  ([Ro], [B-B-HR]). It is a fundamental result of Orlov [Or1, Th. 2.19] that every such equivalence is a **Fourier-Mukai transform**, i.e. there is an object  $\mathcal{A} \in D^b(X \times Y)$ , unique up to isomorphism, called its **kernel**, such that  $F$  is isomorphic to the functor  $\Phi_{\mathcal{A}} := p_*(q^*(-) \otimes \mathcal{A})$ , where  $p_*$ ,  $q^*$  and  $\otimes$  are derived functors. Therefore such pairs  $X$  and  $Y$  are also called **Fourier-Mukai partners**. Orlov also proved the following Theorem and stated the conjecture below.

**Theorem 1.** ([Or2, Th. 1]) *If  $\dim X = \dim Y = n$  and  $\Phi_{\mathcal{A}}: D^b(X) \rightarrow D^b(Y)$  is an exact fully faithful functor satisfying the following condition*

$$(*) \quad \text{the dimension of the support of } \mathcal{A} \in D^b(X \times Y) \text{ is } n,$$

*then the motive  $M(X)_{\mathbb{Q}}$  is a direct summand of  $M(Y)_{\mathbb{Q}}$ . If in addition the functor  $\Phi_{\mathcal{A}}$  is an equivalence then the motives  $M(X)_{\mathbb{Q}}$  and  $M(Y)_{\mathbb{Q}}$  are isomorphic in Voevodsky's triangulated category of motives  $DM_{gm}(\mathbb{C})_{\mathbb{Q}}$ . Moreover the same results hold true at the level of integral motives.*

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**Conjecture 2.** ([Or2, Conj. 1]) *Let  $X$  and  $Y$  be smooth projective varieties and let  $F: D^b(X) \rightarrow D^b(Y)$  be a fully faithful functor. Then the motive  $M(X)_{\mathbb{Q}}$  is a direct summand of  $M(Y)_{\mathbb{Q}}$ . If  $F$  is an equivalence then the motives  $M(X)_{\mathbb{Q}}$  and  $M(Y)_{\mathbb{Q}}$  are isomorphic.*

In [Hu1, 2.7] Huybrechts proved that if  $F: D^b(X) \simeq D^b(X)$  is a self equivalence then it acts identically on cohomology if and only if it acts identically on Chow groups (see section 5). This naturally suggests the following conjecture, which appears in [Hu2, Conj. 3.4].

**Conjecture 3.** *Let  $X$  be a complex K3 surface and let  $f \in \text{Aut}(X)$  be a symplectic automorphism, i.e.  $f^*$  acts as the identity on  $H^{2,0}(X)$ . Then  $f^* = \text{id}$  on  $CH^2(X)$ .*

In section 2 we recall some results on the finite dimensionality of motives and their Chow-Künneth decompositions.

In section 3, after some general remarks on the derived equivalences between two smooth projective varieties  $X$  and  $Y$ , we relate the derived equivalence with ungraded motives and finite-dimensionality (Proposition 15).

In section 4 we specialize to the case of K3 surfaces  $X$  and  $Y$  and prove our main result (Theorem 21): Orlov's conjecture holds true for K3 surfaces  $X$  and  $Y$  if the motive of  $X$  is finite-dimensional and  $X$  admits an elliptic fibration, a case that always occurs if the Picard rank  $\rho(X)$  is at least 5. This restriction can possibly be removed, according to a claimed result by Mukai in [Mu2, Th2].

In section 5 we consider the case of a K3 surface with a symplectic involution  $\iota$  and prove (Theorem 27) that Huybrechts' Conjecture 3 holds true for  $f = \iota$  if  $X$  has a finite-dimensional motive. We also show (Theorem 30 and Examples 31) the existence of K3 surfaces  $X$  and  $Y$  which are not derived equivalent but with isomorphic motives.

**Acknowledgements.** We thank Claudio Bartocci for many helpful comments on a early draft of this paper.

## 2. CATEGORIES OF MOTIVES AND FINITE DIMENSIONALITY

Let  $X$  be a smooth variety over a perfect field  $k$  and let  $CH^i(X)$  be the Chow group of cycles of codimension  $i$  modulo rational equivalence. We will denote by  $A^i(X) = CH^i(X)_{\mathbb{Q}}$  the  $\mathbb{Q}$ -vector space  $CH^i(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ .

**2.1. Pure motives.** Let  $\mathcal{M}_{rat}^{eff}(k)$  be the *covariant* pseudo-abelian, tensor, additive category of **effective Chow motives** with  $\mathbb{Q}$ -coefficients over a perfect field  $k$ . Its objects are couples  $(X, p)$  where  $X$  is a smooth projective variety and  $p \in CH_{\dim X}(X \times X)_{\mathbb{Q}}$  is a projector, i.e.  $p \circ p = p^2 = p$ . Morphisms between  $(X, p)$  and  $(Y, q)$  in  $\mathcal{M}_{rat}^{eff}$  are given by correspondences  $\Gamma \in A_{\dim X}(X \times Y)$ . More precisely:

$$\text{Hom}_{\mathcal{M}_{rat}^{eff}(k)}((X, p), (Y, q)) = q \circ CH_{\dim X}(X \times Y)_{\mathbb{Q}} \circ p.$$

The motive of a smooth projective variety  $X$  is defined as  $\mathfrak{h}(X) = (X, \Delta_X) \in \mathcal{M}_{rat}^{eff}(k)$ , thus giving a covariant monoidal functor  $h: \mathcal{S}mProj/k \rightarrow \mathcal{M}_{rat}^{eff}(k)$  which sends  $f: X \rightarrow Y$  to its graph  $\mathfrak{h}(f) = [\Gamma_f]: \mathfrak{h}(X) \rightarrow \mathfrak{h}(Y)$ . Let  $X = \mathbb{P}^1$ , then the structure map  $X \rightarrow \text{Spec}(k)$  together with the inclusion of a closed point  $P \in \mathbb{P}^1$  (eventually defined over an algebraic extension of  $k$ , see [K-M-P, 7.2.8]) induces a splitting

$$\mathfrak{h}(\mathbb{P}^1) \simeq \mathbb{1} \oplus \mathbb{L}$$

where  $\mathbb{1} = (\text{Spec}(k), \Delta_{\text{Spec}(k)}) \simeq (\mathbb{P}^1, [\mathbb{P}^1 \times P])$  is the unit of the tensor structure and  $\mathbb{L} = (\mathbb{P}^1, [P \times \mathbb{P}^1])$  is the **Lefschetz motive**. By  $\mathcal{M}_{rat}(k)$  we will denote the tensor category of **covariant Chow motives**, obtained from  $\mathcal{M}_{rat}^{eff}(k)$  by inverting  $\mathbf{L}$ , as in [K-M-P].

We will also consider the  $\mathbb{Q}$ -linear rigid tensor category of **ungraded covariant Chow motives**  $\mathcal{UM}_{rat}(k)$  (see for example [Ma, §2, §3, p. 459] and [D-M, 1.3]). It is the pseudo-abelian hull of the  $\mathbb{Q}$ -linear additive category of ungraded correspondences. Hence, its objects are pairs  $(X, e)$  with  $X$  a smooth projective variety,  $e \in CH_*(X \times X)_{\mathbb{Q}} = \bigoplus_{i=0}^{2 \dim X} CH_i(X \times X)_{\mathbb{Q}}$  a projector, and

$$\text{Hom}_{\mathcal{UM}_{rat}(k)}((X, e), (Y, f)) = f \circ CH_*(X \times Y)_{\mathbb{Q}} \circ e;$$

the ungraded motive of  $X$  is  $\mathfrak{h}(X)_{\text{un}} := (X, \Delta_X)$ ; its endomorphism algebra is the  $\mathbb{Z}$ -graded ring (w.r.t. composition of correspondences, see [Ma, §4 p. 452])

$$\text{End}_{\mathcal{UM}_{rat}(k)}(\mathfrak{h}(X)_{\text{un}}) = CH_*(X \times X)_{\mathbb{Q}}.$$

$\mathcal{UM}_{rat}(k)$  is a rigid  $\mathbb{Q}$ -linear tensor category in the obvious way.

**2.2. Mixed motives.** Let  $DM_{gm}^{eff}(k)$  be the triangulated category of **effective geometrical motives** constructed by Voevodsky in [Voev]. We recall that there is a covariant functor  $M: \mathcal{S}m/k \rightarrow DM_{gm}^{eff}(k)$  where  $\mathcal{S}m/k$  is the category of smooth schemes of finite type over  $k$ . We shall write  $DM_{gm}^{eff}(k, \mathbb{Q})$  for the pseudo-abelian hull of the category obtained from  $DM_{gm}^{eff}(k)$  by tensoring morphisms with  $\mathbb{Q}$ , and usually abbreviate it into  $DM_{gm}^{eff}(k)$ . Then  $M$  induces a covariant functor

$$\Phi: \mathcal{M}_{rat}^{eff}(k) \rightarrow DM_{gm}^{eff}(k)$$

which is a full embedding. We will denote by  $DM_{gm}(k) = DM_{gm}(k, \mathbb{Q})$  the category obtained from  $DM_{gm}^{eff}(k)$  by inverting the image  $\mathbb{Q}(1)$  of  $\mathbf{L}$ . Hence, for two smooth projective varieties  $X$  and  $Y$ ,  $\mathfrak{h}(X) \simeq \mathfrak{h}(Y)$  in  $\mathcal{M}_{rat}(k)$  if and only if the images  $M(X)$  and  $M(Y)$  are isomorphic in  $DM_{gm}(k)$ .

**2.3. Finite-dimensional motives.** We now recall several notion of “finiteness” on motives (see [Ki, 3.7], [Maz, 1.3], [An1, 12] and [An2, 3]). Let  $\mathcal{C}$  be a pseudoabelian,  $\mathbb{Q}$ -linear, symmetric tensor category and let  $A$  be an object in  $\mathcal{C}$ . Thanks to the symmetry isomorphism of  $\mathcal{C}$  the symmetric group on  $n$  letters  $\Sigma_n$  acts naturally on the  $n$ -fold tensor product  $A^{\otimes n}$  of  $A$  by itself for each object  $A$ : any  $\sigma \in \Sigma_n$  defines a map  $\sigma_{A^{\otimes n}} : A^{\otimes n} \rightarrow A^{\otimes n}$ . We recall that there is a one-to-one correspondence between all irreducible representations of the group  $\Sigma_n$  (over  $\mathbb{Q}$ ) and all partitions of the integer  $n$ . Let  $V_\lambda$  be the irreducible representation corresponding to a partition  $\lambda$  of  $n$  and let  $\chi_\lambda$  be the character of the representation  $V_\lambda$ , then

$$d_\lambda = \frac{\dim(V_\lambda)}{n!} \sum_{\sigma \in \Sigma_n} \chi_\lambda(\sigma) \cdot \sigma \in \mathbb{Q}\Sigma_n$$

gives, when  $\lambda$  varies among the partitions of  $n$ , a set of pairwise orthogonal central (non primitive) idempotents in the group algebra  $\mathbb{Q}\Sigma_n$ ; the two-sided ideal  $(d_\lambda) = d_\lambda \mathbb{Q}\Sigma_n$  is the isotypic component of  $V_\lambda$  inside  $\mathbb{Q}\Sigma_n$  hence  $(d_\lambda) \cong V_\lambda^\lambda$  as  $\mathbb{Q}\Sigma_n$ -modules. Let

$$d_\lambda^A = \frac{\dim(V_\lambda)}{n!} \sum_{\sigma \in \Sigma_n} \chi_\lambda(\sigma) \cdot \sigma_{A^{\otimes n}} \in \text{Hom}_{\mathcal{C}}(A^{\otimes n}, A^{\otimes n})$$

where  $\sigma_{A^{\otimes n}}$  is the morphism associated to  $\sigma$ . Then  $\{d_\lambda^A\}$  is a set of pairwise orthogonal idempotents in  $\text{Hom}_{\mathcal{C}}(A^{\otimes n}, A^{\otimes n})$  such that  $\sum d_\lambda^A = \text{Id}_{A^{\otimes n}}$ . The category  $\mathcal{C}$  being pseudoabelian, they give a functorial decomposition

$$A^{\otimes n} = \oplus_{|\lambda|=n} S_\lambda(A) \quad (S_\lambda(A) = \text{Im } d_\lambda^A),$$

where  $S_\lambda$  is the **isotypic Schur functor** associated to  $\lambda$  (which is a just “multiple” of the classical one). The  $n$ -th symmetric product  $\text{Sym}^n A$  of  $A$  is then defined to be the image  $\text{Im}(d_\lambda^A)$  when  $\lambda$  corresponds to the partition  $(n)$ , and the  $n$ -th exterior power  $\wedge^n A$  is  $\text{Im}(d_\lambda^A)$  when  $\lambda$  corresponds to the partition  $(1, \dots, 1)$ . If  $\mathcal{C} = \mathcal{M}_{\text{rat}}(k)$  and  $A = \mathfrak{h}(X) \in \mathcal{M}_{\text{rat}}(k)$  for a smooth projective variety  $X$ , then  $\wedge^n A$  is the image of  $\mathfrak{h}(X^n) = \mathfrak{h}(X)^{\otimes n}$  under the projector  $(1/n!)(\sum_{\sigma \in \Sigma_n} \text{sgn}(\sigma) \Gamma_\sigma)$ , while  $\text{Sym}^n A$  is its image under the projector  $(1/n!)(\sum_{\sigma \in \Sigma_n} \Gamma_\sigma)$ .

**Definition 4.** The object  $A$  in  $\mathcal{C}$  is said to be **Schur finite** if  $S_\lambda(A) = 0$  for some partition  $\lambda$  (i.e.  $d_\lambda^A = 0$  in  $\text{End}_{\mathcal{C}}(A^{\otimes n})$ ); it is said to be **evenly (oddly) finite-dimensional** if  $\wedge^n A = 0$  ( $\text{Sym}^n A = 0$ ) for some  $n$ . An object  $A$  is **finite-dimensional** (in the sense of Kimura and O’Sullivan) if it can be decomposed into a direct sum  $A_+ \oplus A_-$  where  $A_+$  is evenly finite-dimensional and  $A_-$  is oddly finite-dimensional.

If  $A$  is evenly and oddly finite-dimensional then  $A = 0$  (see [Ki, 6.2] and [An2, 6.2]).

**Remark 5.** From the definition it follows that, for a smooth projective variety  $X$  over  $k$ , the motive  $\mathfrak{h}(X)$  is finite-dimensional in  $\mathcal{M}_{\text{rat}}(k)$  if and only if  $M(X)$  is finite-dimensional in  $DM_{\text{gm}}(k)$ .

Kimura's nilpotence Theorem in [Ki, 7.5] says that if  $M$  is finite-dimensional, any numerically trivial endomorphism **universally of trace zero** (i.e. given by a correspondence which is numerically trivial as an algebraic cycle) of  $M$  is nilpotent; therefore

**Theorem 6.** (*Kimura*) *If  $M$  and  $N$  are two finite-dimensional Chow motives and  $f: M \rightarrow N$  is a morphism, then  $f$  is an isomorphism if and only if its reduction modulo numerical equivalence is such (see [An2 3.16.2]).*

In particular, if  $M \in \mathcal{M}_{\text{rat}}$  is a finite-dimensional motive such that  $H^*(M) = 0$ , where  $H^*$  is any Weil cohomology, then  $M = 0$  ([Ki, 7.3]).

**Remark 7.** For Schur-finite objects such a nilpotency result holds only under some extra assumptions as shown in [DP-M1] and [DP-M2], but not in general. In fact let  $\mathcal{C}$  be the  $\mathbb{Q}$ -linear rigid tensor category of bounded chain complexes of finitely generated  $\mathbb{Q}$ -vector spaces with the usual tensor structure and the “Koszul” commutativity constraint. Then  $\text{Id}_{\mathbb{Q}}: \mathbb{Q} \rightarrow \mathbb{Q}$  can be thought of as an object  $A$  of  $\mathcal{C}$ , concentrated in homological degrees 1 and 0. It is indecomposable as  $\text{End}_{\mathcal{C}}(A) \cong \mathbb{Q}$ , and it is not finite-dimensional for  $\wedge^n(A) \neq 0$  and  $\text{Sym}^n(A) \neq 0$  (as complexes) for each  $n \in \mathbb{N}$ . On the other hand  $S_{(2,2)}(A) = 0$ , i.e.  $A$  is Schur-finite, for it is so under the obvious faithful (but not full)  $\mathbb{Q}$ -linear tensor functor towards  $\mathbb{Z}/2$ -graded  $\mathbb{Q}$ -vector spaces. Moreover, due to the Koszul rule,  $\text{Id}_A$  is universally of trace zero but not nilpotent.

### Examples 8.

(1) Finite-dimensionality and Schur-finiteness are stable under direct sums, tensor products, and direct summand. More precisely:  $S_{\lambda}(B) = 0$  whenever  $B$  is a direct summand of  $A$  with  $S_{\lambda}(A) = 0$ . It is also true that a direct summand of a finite-dimensional object is such ([An2, 3.7]). Finite-dimensionality implies Schur-finiteness, but the converse does not hold not even in  $DM_{gm}(k)$ . In fact Peter O’Sullivan showed that there exist smooth surfaces  $S$  whose motives in  $DM_{gm}(k)$  is Schur-finite but not finite dimensional, see [Maz, 5.11].

(2) Clearly we have  $\wedge^2 \mathbb{1} = 0$  in any *symmetric* tensor category. It is also straightforward that  $\wedge^2 \mathbb{L} = 0$  for the Lefschetz motive, and  $\wedge^3 \mathfrak{h}(\mathbb{P}^1) = 0$ . Kimura showed  $\text{Sym}^{2g+1}(\mathfrak{h}^1(C)) = 0$  for any smooth projective curve  $C$  of genus  $g$  [Ki, 4.2].

We also have Kimura’s conjecture:

**Conjecture 9.** *Any motive in  $\mathcal{M}_{\text{rat}}$  is finite-dimensional.*

**Remark 10.** The status of the conjecture is the following.

(1) The conjecture is true for curves, abelian varieties, Kummer surfaces, complex surfaces not of general type with  $p_g = 0$  (e.g. Enriques surfaces), Fano 3-folds [G-G]. For a complex surface  $X$  of general type with  $p_g(X) = 0$  the finite-dimensionality of the motive  $\mathfrak{h}(X)$  is equivalent to Bloch’s conjecture, i.e. to the vanishing of the Albanese Kernel of  $X$  (see [G-P, Th. 7]). If the conjecture holds for  $\mathfrak{h}(X)$  then it holds

true for  $\mathfrak{h}(Y)$  with  $Y$  a smooth projective variety dominated by  $X$ . The full subcategory of  $\mathcal{M}_{\text{rat}}$  on finite-dimensional objects is a  $\mathbb{Q}$ -linear rigid tensor subcategory closed under direct summand.

(2) Let  $X$  be a K3 surface; then  $\mathfrak{h}(X)$  is finite-dimensional in the following cases, see [Pe3]

- $\rho(X) = 19$  or  $\rho(X) = 20$ . In these cases  $X$  has a *Nikulin involution* which gives a *Shioda-Inose structure*, in the sense of [Mo, 6.1], and the transcendental motive  $\mathfrak{t}_2(X)$  of  $X$  (see 2.4) is isomorphic to the transcendental motive of a *Kummer surface* [Pe3, Th. 4].
- $X$  has a non-symplectic group  $G$  acting trivially on the algebraic cycles and the order of the kernel (a finite group) of the map  $\text{Aut}(X) \rightarrow \mathcal{O}(\text{NS}(X))$  is different from 3, where  $\mathcal{O}(\text{NS}(X))$  is the group of isometries of  $\text{NS}(X)$ . Then, by a result in [L-S-Y, Th. 5],  $X$  is dominated by a *Fermat surface*  $F_n$ , whose motive is of abelian type (hence finite-dimensional) by the *Shioda-Katsura inductive structure* [S-K, Th. I]. K3 surfaces satisfying these conditions have  $\rho(X) = 2, 4, 6, 10, 12, 16, 18, 20$ .

By a result of Deligne ([De, 6.4]), for every complex polarized K3 surface there exists a smooth family of polarized K3 surfaces  $\{X\}_{t \in \Delta}$ , with  $\Delta$  the unit disk, such that the central fibre  $X_0$  is isomorphic to  $X$ . Therefore the finite-dimensionality of the motive of a general K3 surface, i.e. with  $\rho(X) = 1$ , implies the finite-dimensionality of the motive of any K3 surface, see [Pe1, 4.3].

(3) In all the known cases where the motive  $\mathfrak{h}(X)$  is finite-dimensional, it lies in the tensor subcategory of  $\mathcal{M}_{\text{rat}}(k)$  generated by the motives of abelian varieties (see [An, 2.5]).

The following result will appear in [DP].

**Proposition 11.** *Let  $M = (X, p)$  be an effective Chow motive. Then*

(a) *The (graded) motive  $M$  is Schur-finite if and only if the ungraded motive  $M_{\text{un}}$  is such. More precisely for any partition  $\lambda$  we have  $S_{\lambda}^{\mathcal{M}_{\text{rat}}(k)}(M) = 0$  if and only if  $S_{\lambda}^{\mathcal{UM}_{\text{rat}}(k)}(M_{\text{un}}) = 0$ . In particular, being  $M$  even or odd depends only on the ungraded isomorphism class of the ungraded motive  $M_{\text{un}}$ .*

(b) *If  $M$  is finite-dimensional in  $\mathcal{M}_{\text{rat}}(k)$  then  $M_{\text{un}}$  is so in  $\mathcal{UM}_{\text{rat}}(k)$ . Moreover, if  $M = \mathfrak{h}(X)$  with  $X$  a variety such that the projections on the even and the odd part of the cohomology (w.r.t. a given Weil cohomology theory) are algebraic then  $\mathfrak{h}(X)$  is finite-dimensional if and only if  $\mathfrak{h}(X)_{\text{un}}$  is.*

**Remark 12.** The hypothesis in (b) of Proposition 11 is Jannsen's *homological sign conjecture*  $C^+(X)$  [An2, 5.1.3], called  $S(X)$  in [Ja, 13.3].

**2.4. The refined Chow-Künneth decomposition.** Let for simplicity  $k = \mathbb{C}$  in what follows. We recall from [K-M-P, 2.1] that the covariant Chow motive  $\mathfrak{h}(S) \in \mathcal{M}_{rat}(\mathbb{C})_{\mathbb{Q}}$  of any smooth projective surface  $S$  has a **refined Chow-Künneth decomposition**

$$\sum_{0 \leq i \leq 4} \mathfrak{h}_i(S)$$

corresponding to a splitting  $\Delta_S = \sum_{0 \leq i \leq 4} \pi_i$  of the diagonal in  $H^*(S \times S)$ . Here  $\mathfrak{h}_0(S) = (S, [S \times P]) \simeq (\text{Spec}(\mathbb{C}), \text{Id}) = \mathbb{1}$  and  $\mathfrak{h}_4(S) = (S, [P \times S]) \simeq \mathbb{L}^2$ , where  $P$  is a rational point on  $S$ . Also

$$\mathfrak{h}_2(S) = \mathfrak{h}_2^{\text{alg}}(S) \oplus \mathfrak{t}_2(S)$$

with  $\mathfrak{h}_2^{\text{alg}}(S) = (S, \pi_2^{\text{alg}})$  the effective Chow motive defined by the idempotent

$$\pi_2^{\text{alg}}(S) = \sum_{1 \leq h \leq \rho} \frac{[D_h \times D_h]}{D_h^2} \in A_2(S \times S)$$

where  $\rho = \rho(S)$  is the rank of the Neron-Severi  $\text{NS}(S)$  and  $\{D_h\}$  is an orthogonal bases of  $\text{NS}(S)_{\mathbb{Q}}$ . It follows that  $\mathfrak{h}_2^{\text{alg}}(S) \simeq \mathbb{L}^{\oplus \rho}$ .

**Definition 13.** The Chow motive  $\mathfrak{t}_2(S) = (S, \pi_2^{\text{tr}}, 0)$ , with  $\pi_2^{\text{tr}} = \pi_2 - \pi_2^{\text{alg}}$ , is the **transcendental part** of the motive  $\mathfrak{h}(S)$ . Then  $H^i(\mathfrak{t}_2(S)) = 0$  if  $i \neq 2$  and  $H^2(\mathfrak{t}_2(S)) = H_{\text{tr}}^2(S) = \pi_2^{\text{tr}} H^2(S, \mathbb{Q}) = H_{\text{tr}}^2(S, \mathbb{Q})$ .

The Chow motive  $\mathfrak{t}_2(S)$  does not depend on the choices made to define the refined Chow-Künneth decomposition, it is functorial on  $S$  for the action of correspondences, and it is a *birational invariant* of  $S$  (see [K-M-P]).

**Remark 14.** For any smooth projective surface  $S$ , all the motives  $\mathfrak{h}_i(S)$  appearing in a refined Chow-Künneth decomposition, except possibly for  $\mathfrak{t}_2(S)$  are finite dimensional. Therefore the motive  $\mathfrak{h}(S)$  of a surface  $S$  is finite dimensional if and only if the motive  $\mathfrak{t}_2(S)$  is evenly finite dimensional, i.e.  $\wedge^n \mathfrak{t}_2(S) = 0$  for some  $n$ . If  $S$  has no irregularity (i.e.  $q(S) := \dim H^1(S, \mathcal{O}_S) = 0$ ) then  $\mathfrak{h}_1(S) = \mathfrak{h}_3(S) = 0$ .

**2.5. Refined C-K decomposition of a K3 surface.** Let now  $S$  be a smooth (irreducible) projective K3 surface over  $\mathbb{C}$ . As  $S$  is a regular surface (i.e.  $q(S) = 0$ ), its refined Chow-Künneth decomposition has the following shape

$$\mathfrak{h}(S) = \mathbb{1} \oplus \mathfrak{h}_2^{\text{alg}}(S) \oplus \mathfrak{t}_2(S) \oplus \mathbb{L}^{\otimes 2} \simeq \mathbb{1} \oplus \mathbb{L}^{\oplus \rho} \oplus \mathfrak{t}_2(S) \oplus \mathbb{L}^{\otimes 2}$$

with  $1 \leq \rho \leq 20$ . Moreover

$$A_i(\mathfrak{t}_2(S)) = \pi_2^{\text{tr}} A_i(S) = 0 \text{ for } i \neq 0; \quad A_0(\mathfrak{t}_2(S)) = A_0(S)_0,$$

where the last  $\mathbb{Q}$ -vector space is the group of 0-cycles of degree 0 tensored with  $\mathbb{Q}$ . We also have

$$\dim H^2(S) = b_2(S) = 22; \quad \dim H_{\text{tr}}^2(S) = b_2(S) - \rho(S) = 22 - \rho.$$

By  $T_{S,\mathbb{Q}} = H_{\text{tr}}^2(S, \mathbb{Q})$  we will denote the *lattice of transcendental cycles*, tensored with  $\mathbb{Q}$ , it coincides with the orthogonal complement to the Neron-Severi  $\text{NS}(S) \otimes \mathbb{Q}$  in  $H^2(S, \mathbb{Q})$ .

### 3. DERIVED EQUIVALENCE AND MOTIVES

Let  $X$  and  $Y$  be smooth projective varieties over  $\mathbb{C}$ . If  $X$  and  $Y$  are derived equivalent then (see e.g. [Ro], [Hu], [B-B-HR])  $\dim X = \dim Y$ ,  $\kappa(X) = \kappa(Y)$  (where  $\kappa$  is the Kodaira dimension), and  $H^*(X, \mathbb{Q}) \simeq H^*(Y, \mathbb{Q})$  (isomorphism of  $\mathbb{Z}/2$ -graded vector spaces). If  $\dim X = 2$  the surfaces  $X$  and  $Y$  have the same Picard number and the same topological Euler number; and  $X$  is a K3 surface, respectively an abelian surface, if and only if  $Y$  is.

Kawamata conjectured that, up to isomorphism,  $X$  has only a finite number of Fourier-Mukai partners  $Z$  [Ka]. This conjecture is true for curves (and in this case  $Z \simeq X$ , [B-B-HR, 7.16]), surfaces ([B-M]), abelian varieties (see [Ro, 3] and [H-NW, 0.4]), and varieties with ample or antiample canonical bundle, in which case  $Z \simeq X$  (due to Bondal-Orlov, see [B-B-HR, 2.51]).

The following result is somewhat in the same spirit, with respect to the relation between derived equivalence of smooth projective varieties and their associated Chow motives.

**Proposition 15.** *Let  $\Phi_{\mathcal{A}}: D^b(X) \longrightarrow D^b(Y)$  an exact equivalence, then*

**(a)** *The ungraded Chow motives  $\mathfrak{h}(X)_{\text{un}}$  and  $\mathfrak{h}(Y)_{\text{un}}$  are isomorphic. If the condition (\*) in Theorem 1 is satisfied then the isomorphism is given by a correspondence of degree zero, hence  $\mathfrak{h}(X)$  and  $\mathfrak{h}(Y)$  are isomorphic as Chow motives.*

**(b)** *The (graded) motive  $\mathfrak{h}(X)$  is Schur-finite if and only if  $\mathfrak{h}(Y)$  is such.*

**(c)** *If  $X$  is curve, a surface, an abelian variety, or a finite product of them (or any variety if  $k$  is algebraic over a finite field), then  $\mathfrak{h}(X)$  is finite-dimensional if and only if  $\mathfrak{h}(Y)$  is such.*

*Proof.* **(a)** The argument in [Or 1, p. 1243], which we briefly recall can be used to prove that  $\mathfrak{h}(X)_{\text{un}} \cong \mathfrak{h}(Y)_{\text{un}}$  in  $\mathcal{UM}_{\text{rat}}(k)$ . Let  $\mathcal{B} \in D^b(X \times Y)$  be the kernel of the quasi-inverse of  $\Phi_{\mathcal{A}}$ . Using Huybrechts' notation ([Hu1, p. 1534] and [Hu2, 4.1]), we then have (non homogeneous,  $\mathbb{Q}$ -linear) algebraic cycles

$$a = v^{\text{CH}}(\mathcal{A}) := \text{ch}(\mathcal{A}) \cdot \sqrt{\text{td}_{X \times Y}} = p_1^* \left( \sqrt{\text{td}_X} \right) \cdot \text{ch}(\mathcal{A}) \cdot p_2^* \left( \sqrt{\text{td}_Y} \right) \in CH_*(X \times Y)_{\mathbb{Q}},$$

and

$$b = v^{\text{CH}}(\mathcal{B}) = p_1^* \left( \sqrt{\text{td}_Y} \right) \cdot \text{ch}(\mathcal{B}) \cdot p_2^* \left( \sqrt{\text{td}_X} \right) \in CH_*(Y \times X)_{\mathbb{Q}},$$

where  $\text{td}$  is the Todd class and  $\text{ch}: D^b(Z) \longrightarrow CH_*(Z)_{\mathbb{Q}}$  is the composition of the Chern character with the Euler characteristic  $\chi(\mathcal{E}) = \sum (-1)^i [\mathcal{H}^i(\mathcal{E})] \in K_0(Z)$  of the complex of sheaves  $\mathcal{E}$ . Orlov proved, by Grothendieck-Riemann-Roch, that

$$b \circ a = [\Delta_X] = \text{Id}_{\mathfrak{h}(X)_{\text{un}}}, \quad \text{and} \quad a \circ b = [\Delta_Y] = \text{Id}_{\mathfrak{h}(Y)_{\text{un}}}$$



as (ungraded) correspondences.

In case the kernel  $\mathcal{A}$  satisfies the hypothesis  $(*)$  of Theorem 1, that is  $\dim \operatorname{supp}(\mathcal{A}) = \dim X$ , it turns out that the “middle components”  $a_d \in CH_d(X \times Y)_{\mathbb{Q}}$  and  $b_d \in CH_d(Y \times X)_{\mathbb{Q}}$  of the above cycles  $a$  and  $b$  (which are correspondences of degree zero) give an isomorphism at the level of usual Chow motives.

(b) As already observed in Proposition 11, being Schur-finite for a graded motive  $M$  can be tested on  $M_{\text{un}}$ .

(c) In all these cases  $C^+(X)$  holds true, hence Proposition 11 (b) applies.  $\square$

**Example 16.** Let  $X = A$  be an abelian variety,  $Y = \widehat{A}$  its dual and let  $\mathcal{A} = \mathcal{P}_A \in \operatorname{Pic}(A \times \widehat{A})$  be the sheaf complex given by the Poincaré bundle. The corresponding isomorphism of ungraded Chow motives is given by

$$\operatorname{ch}(\mathcal{P}_A): h(A)_{\text{un}} \longrightarrow h(\widehat{A})_{\text{un}}$$

because the Todd classes are 1 for abelian varieties. It can be shown (see [B-L 16.3]) that it coincides with the *motivic Fourier-Mukai transform* of Deninger and Murre ([D-M, 2.9]). We note that in this case the dimension of the support of  $\mathcal{A}$  is equal to  $\dim(A \times \widehat{A}) = 2 \cdot \dim A$ . As  $A$  and  $\widehat{A}$  are isogenous it follows that their Chow motives (with  $\mathbb{Q}$ -coefficients) are isomorphic (see for example [An1, 4.3.3]).

**Remarks 17.** Let us make two comments on Orlov’s hypothesis  $(*)$ , that is “the dimension of the support of the kernel  $\mathcal{A}$  of the equivalence  $D^b(X) \simeq D^b(Y)$  equals  $\dim X$ ”.

(1) If  $\Phi_{\mathcal{A}}$  is an equivalence then the natural projections

$$\operatorname{supp}(\mathcal{A}) \longrightarrow X, \quad \operatorname{supp}(\mathcal{A}) \longrightarrow Y$$

are surjective [Hu, 6.4]. Therefore, in general,  $\dim \operatorname{supp}(\mathcal{A}) \geq \dim X$  whenever  $\Phi_{\mathcal{A}}$  is an equivalence.

(2) If  $\Phi_{\mathcal{A}}$  is an equivalence and Orlov’s hypothesis  $(*)$  holds true then  $X$  and  $Y$  are **K-equivalent**, a notion due to Kawamata [Ka] (see [B-B-HR, 2.48]). In case  $X$  and  $Y$  are smooth projective complex *surfaces*, they are K-equivalent if and only if they are isomorphic [B-B-HR, 7.19]. This is, in general, not the case for K3 surfaces, see for example [So].

In connection with the result in [Or2, Th. 1] Orlov made the following more precise conjecture [Or2, Conj. 2]:

**Conjecture 18.** *Let  $\mathcal{A}$  be an object on  $X \times Y$  for which  $\Phi_{\mathcal{A}}: D^b(X) \longrightarrow D^b(Y)$  is an equivalence. Then there are line bundles  $L$  and  $M$  on  $X$  and  $Y$ , respectively, such that the  $\dim X$  component of the cycle associated to  $\mathcal{A}' := p_1^*L \otimes \mathcal{A} \otimes p_2^*M$  determines an isomorphism between the motives  $M(X)_{\mathbb{Q}}$  and  $M(Y)_{\mathbb{Q}}$  in  $DM_{gm}(\mathbb{C})_{\mathbb{Q}}$ .*

## 4. DERIVED EQUIVALENCE AND COMPLEX K3 SURFACES

Let us now consider Orlov's Conjecture 2 in low dimension; a case of particular interest is that of K3 surfaces. We recall that if  $Y$  is a Fourier-Mukai partner of a K3 surface  $X$  (respectively abelian surface), then also  $Y$  is a K3 surface (respectively abelian surface).

We fix some notation. For a K3, or abelian, smooth projective complex surface  $X$  we have the **Mukai lattice**, also called **extended Hodge lattice** in [B-M, 5], that is the cohomology ring

$$\tilde{H}(X, \mathbb{Z}) := H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}),$$

endowed with the symmetric bilinear form

$$\langle (r_1, D_1, s_1), (r_2, D_2, s_2) \rangle := D_1 \cdot D_2 - r_1 s_2 - r_2 s_1,$$

and the following Hodge decomposition

$$\begin{aligned} \tilde{H}^{(0,2)}(X, \mathbb{C}) &= H^{0,2}(X, \mathbb{C}), & \tilde{H}^{(2,0)}(X, \mathbb{C}) &= H^{2,0}(X, \mathbb{C}), \\ \tilde{H}^{(1,1)}(X, \mathbb{C}) &= H^0(X, \mathbb{C}) \oplus H^{1,1}(X, \mathbb{C}) \oplus H^4(X, \mathbb{C}). \end{aligned}$$

Inside  $H^2(X, \mathbb{Z})$  we have two sublattices, the **Neron-Severi lattice**

$$\text{NS}(X) = H^2(X, \mathbb{Z}) \cap H^{1,1}(X, \mathbb{C}),$$

and its orthogonal complement  $T_X$ , the **transcendental lattice** of  $X$ . The transcendental lattice inherits a Hodge structure from  $H^2(X, \mathbb{Z})$ .

**Definition 19.** Let  $X$  and  $Y$  be two complex K3 surfaces. A map  $T_X \rightarrow T_Y$  (resp.  $T_{X, \mathbb{Q}} \rightarrow T_{Y, \mathbb{Q}}$ ) is a **Hodge homomorphism of** (resp. **rational**) **Hodge structures** if it preserves the Hodge structures of  $H_{tr}^2(X) \otimes \mathbb{C}$  and of  $H_{tr}^2(Y) \otimes \mathbb{C}$ , i.e. if the one dimensional subspace  $H^{2,0}(X) \subset T_X \otimes \mathbb{C}$  goes to  $H^{2,0}(Y) \subset T_Y \otimes \mathbb{C}$ . A Hodge isomorphism  $T_X \rightarrow T_Y$  is an **Hodge isometry** if it is an isometry with respect to the quadratic form induced by the usual intersection form. A rational Hodge isometry  $\phi: T_{X, \mathbb{Q}} \rightarrow T_{Y, \mathbb{Q}}$  is **induced by an algebraic cycle**  $\Gamma \in CH_2(X \times Y)_{\mathbb{Q}}$  if  $\phi = \Gamma_*: T_{X, \mathbb{Q}} \rightarrow T_{Y, \mathbb{Q}}$  (cf. [Mu, pp. 346-347]).

Due to work of Mukai and Orlov ([Mu], [Or1, 3.3 and 3.13], [B-M, 5.1]) we have the following result:

**Theorem 20.** *Let  $X$  and  $Y$  be a pair of K3 (resp. abelian) surfaces. The following statements are equivalent.*

- (a)  *$X$  and  $Y$  are derived equivalent,*
- (b) *the transcendental lattices  $T_X$  and  $T_Y$  are Hodge isometric,*
- (c) *the extended Hodge lattices  $\tilde{H}(X, \mathbb{Z})$  and  $\tilde{H}(Y, \mathbb{Z})$  are Hodge isometric,*
- (d)  *$Y$  is isomorphic to a fine, two-dimensional moduli space of stable sheaves on  $X$ .*

The next result relates the finite-dimensionality of the motive of a K3 surface with Orlov's conjecture.

**Theorem 21.** *Let  $X, Y$  be smooth projective K3 surfaces over  $\mathbb{C}$  such that  $X$  has an elliptic fibration and the Chow motive  $\mathfrak{h}(X)$  is finite dimensional. If  $D^b(X) \simeq D^b(Y)$  then the motives  $M(X)$  and  $M(Y)$  are isomorphic in  $DM_{gm}(\mathbb{C})$ .*

*Proof.* By point (b) of Proposition 15 we know that  $\mathfrak{h}(Y)$  is finite-dimensional. Theorem 20 ensures the existence of a Hodge isometry  $\phi : T_{X,\mathbb{Q}} \xrightarrow{\sim} T_{Y,\mathbb{Q}}$  which, by [Ni, Th. 3], is induced by an algebraic cycle, i.e. there exists an algebraic correspondence  $\Gamma \in CH_2(X \times Y)_{\mathbb{Q}}$  such that  $\Gamma_* = \phi$ . Then  $\pi_2^Y \circ \Gamma \circ \pi_2^X$  induces an isomorphism between the transcendental motives as homological motives, hence numerical ones; thus, thanks to Theorem 6, it is an isomorphism at the level of Chow motives by finite-dimensionality. Then  $\mathfrak{h}(X)$  and  $\mathfrak{h}(Y)$  are isomorphic in  $\mathcal{M}_{rat}(\mathbb{C})$ , hence  $M(X)$  and  $M(Y)$  are isomorphic in  $DM_{gm}(\mathbb{C})$ .  $\square$

**Remark 22.** Besides the properties of finite-dimensional objects, the other key point in the previous argument is the algebraicity of  $\phi$ . This question goes back to a **Săfărevič's conjecture** stated at the ICM 1970 in Nice [Sh, B4 p. 416]. Shioda and Inose verified the conjecture in [S-I] for singular K3 surfaces (those having the maximum possible Picard number, i.e.  $\rho(X) = 20$ ). Then Mukai proved it in [Mu1, 1.10] for K3 surfaces with  $\rho(X) \geq 11$ , and Nikulin showed its validity in [Ni, proof of Th.3] whenever  $NS(X)$  contains a (nonzero) square zero element; this is certainly the case if  $\rho \geq 5$  and, according to Pjatetskii-Săpiro and Săfărevič [PS-S], it is equivalent to the existence of an elliptic fibration on  $X$ . Eventually Mukai claimed to have completely solved the problem at ICM 2002 in Beijing [Mu2, Th. 2], hence the hypothesis on the elliptic fibration could be removed.

## 5. NIKULIN INVOLUTIONS

Let  $X$  be a smooth projective K3 surface over  $\mathbb{C}$  and let  $\Phi_{\mathcal{A}} : D^b(X) \xrightarrow{\sim} D^b(X)$  be an autoequivalence. To  $\Phi_{\mathcal{A}}$  we can associate an Hodge isometry

$$\Phi_{\mathcal{A}}^H : \tilde{H}(X, \mathbb{Z}) \simeq \tilde{H}(X, \mathbb{Z}),$$

as well as an automorphism of the Chow group

$$\Phi_{\mathcal{A}}^{CH} : CH^*(X) \simeq CH^*(X)$$

induced by the correspondence  $v^{CH}(\mathcal{A}) \in CH^*(X \times X)$  defined in [Hu2, 4.1]. We therefore get the two representations

$$\begin{array}{ccc} & & \text{Aut}(CH^*(X)) \\ & \nearrow^{\rho^{CH}} & \\ \text{Aut}(D^b(X)) & & \\ & \searrow_{\rho^H} & \\ & & \mathcal{O}(\tilde{H}(X, \mathbb{Z})) \end{array}$$

Here  $\mathcal{O}(\tilde{H}(X, \mathbb{Z}))$  is the group of all integral Hodge isometries of the weight two Hodge structure defined on the Mukai lattice  $\tilde{H}(X, \mathbb{Z})$  and  $\text{Aut}(CH^*(X))$  denotes the group of all automorphisms of the additive group  $CH^*(X)$ . The following Theorem has been proved by D. Huybrechts in [Hu1, 2.7].

**Theorem 23.**  $\text{Ker}(\rho^H) = \text{Ker}(\rho^{CH})$ .

From Theorem 23, if  $\rho^H(\Phi_{\mathcal{A}}) = \Phi_{\mathcal{A}}^H$  is the identity in  $\mathcal{O}(\tilde{H}(X, \mathbb{Z}))$ , then the correspondence  $v^{CH}(\mathcal{A})$  acts as the identity on  $CH^*(X)$ . In particular  $\phi_{\mathcal{A}}^H$  acts as the identity on  $H^{2,0}(X) \simeq H^0(X, \Omega_X^2) \subset H_{tr}^2(X, \mathbb{C})$ . The above Theorem suggested Huybrechts' conjecture 3, that is that any *symplectic* automorphism  $f \in \text{Aut}(X)$  acting trivially on  $H^{2,0}(X)$  acts trivially also on  $CH^2(X)$ .

In this section we deal with the case of a *symplectic involution*.

**Definition 24.** A **Nikulin involution**  $\iota$  on a K3 surface  $X$  is a symplectic involution, i.e.  $\iota^*(\omega) = \omega$  for all  $\omega \in H^{2,0}(X)$ .

A Nikulin involution  $\iota$  on a complex projective K3  $X$  has the following special properties, as proved by Nikulin (see e.g. [Mo, 5.2]):

- the fixed locus of  $\iota$  consists of precisely eight distinct points and
- the minimal resolution  $Y$  of the quotient  $X/\iota = X / \langle \iota \rangle$  is a K3 surface.

The surface  $Y$  can also be obtained as the quotient of the blow up  $\tilde{X}$  of  $X$  in the 8 fixed points by the extension  $\tilde{\iota}$  of  $\iota$  to  $\tilde{X}$  ([Mo, 3], [VG-S, 1.4]). In other words we get the commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{b} & X \\ g \downarrow & & \downarrow \\ Y & \longrightarrow & X/\iota \end{array}$$

where  $Y$  is a desingularization of the quotient surface  $X/\iota$  and  $Y \simeq \tilde{X}/\tilde{\iota}$ , with  $\tilde{\iota}$  the involution induced by  $i$  on  $\tilde{X}$ .

As explained in [VG-S, 2.1] a K3 surface with a Nikulin involution has  $\rho(X) \geq 9$ . Moreover ([VG-S, 2.4])  $\iota$  induces an isomorphism  $\phi_\iota: T_{X,\mathbb{Q}} \xrightarrow{\sim} T_{Y,\mathbb{Q}}$  of rational Hodge structures.

Let  $X$ ,  $\tilde{X}$ , and  $Y$  be as in the diagram above and let  $\mathbf{t}_2(X)$  be the transcendental part of the motive of  $X$ . By [Ma, §3 Example 1] the degree 2 map  $g$  induces a splitting in  $\mathcal{M}_{rat}(\mathbb{C})$

$$\mathbf{h}(\tilde{X}) = (X, p) \oplus (X, \Delta_X - p) \simeq \mathbf{h}(Y) \oplus (X, \Delta_X - p)$$

where  $p = 1/2(\Gamma_g^t \circ \Gamma_g) \in A_2(X \times X)$ . Since  $\mathbf{t}_2(-)$  is a birational invariant we have  $\mathbf{t}_2(X) = \mathbf{t}_2(\tilde{X})$ . From the above splitting it follows that  $\mathbf{t}_2(Y)$  is a direct summand of  $\mathbf{t}_2(X)$ , i.e.  $\mathbf{t}_2(X) = \mathbf{t}_2(Y) \oplus N$ .

**Proposition 25.** *Let  $X$ ,  $\tilde{X}$ , and  $Y$  be as in the diagram above. Then*

$$\mathbf{t}_2(X) \simeq \mathbf{t}_2(Y) \iff A_0(X)^\iota = A_0(X)$$

*i.e. if and only if the involution  $\iota$  acts as the identity on  $A_0(X)$ . If  $\mathbf{t}_2(X) \simeq \mathbf{t}_2(Y)$ , then the rational map  $X \rightarrow Y$  induces an isomorphism between the motives  $\mathbf{h}(X)$  and  $\mathbf{h}(Y)$  and therefore also between  $M(X)$  and  $M(Y)$  in  $DM_{gm}(\mathbb{C})$ .*

*Proof.* Let  $k(X)$  be the field of rational functions of  $X$ ; then the Chow group of 0-cycles on  $X_{k(X)}$  may be identified with

$$\lim_{U \subset X} A^2(U \times X) \simeq A_0(X_{k(X)})$$

where  $U$  runs among the open sets of  $X$  (see [Bl, Lecture 1. Appendix]). Since  $\text{Alb}(X) = 0$ , the Albanese kernel  $T(X_{k(X)})$  coincides with  $A_0(X_{k(X)})_0$ . By [K-M-P, 5.10] there is an isomorphism

$$\text{End}_{\mathcal{M}_{rat}}(\mathbf{t}_2(X)) \simeq \frac{A_0(X_{k(X)})}{A_0(X)}$$

where the identity map of  $\mathbf{t}_2(X)$  corresponds to the class of  $[\xi]$  in  $\frac{A_0(X_{k(X)})}{A_0(X)}$ . Here  $\xi$  denotes the generic point of  $X$  and  $[\xi]$  its class as a cycle in  $A_0(X_{k(X)})$ . The involution  $\iota$  induces an involution  $\bar{\iota}$  on  $A_0(X_{k(X)})$ . The splitting

$$[\xi] = 1/2([\xi] + \bar{\iota}([\xi])) + 1/2([\xi] - \bar{\iota}([\xi]))$$

in  $A_0(X_{k(X)})$  corresponds to the splitting of the identity map of  $\mathbf{t}_2(X)$  in  $\mathbf{t}_2(X) = \mathbf{t}_2(Y) \oplus N$ . Therefore  $N = 0$  if and only if  $\bar{\iota}([\xi]) = [\xi]$ . From the equalities  $A_0(\mathbf{t}_2(X)) = A_0(X)_0$ ,  $A_0(\mathbf{t}_2(Y)) = A_0(Y)_0$  and  $A_0(X)^\iota = A_0(Y)$  we get

$$\mathbf{t}_2(X) \simeq \mathbf{t}_2(Y) \iff N = 0 \iff \bar{\iota}([\xi]) = [\xi] \iff A_0(X)^\iota = A_0(X).$$

The rest follows from 2.5 because  $X$  and  $Y$  are K3 surfaces, with  $\rho(X) = \rho(Y)$ .  $\square$

Next we show that for every K3 surface with a Nikulin involution  $\iota$  the finite dimensionality of  $h(X)$  implies that  $\iota$  acts as the identity on  $A_0(X)$ . Therefore for such  $X$  Conjecture 3 holds true.

**Lemma 26.** *Let  $X$  be a K3 surface with a Nikulin involution  $\iota$ . Then  $\rho(X) = \rho(Y)$  and  $t = 6$ , where  $t$  denotes the trace of the involution  $\iota$  on  $H^2(X, \mathbb{C})$ .*

*Proof.* Let  $X$  be a smooth projective surface over  $\mathbb{C}$  with  $q(X) = 0$  and an involution  $\sigma$  and let  $Y$  be a desingularization of  $X/\sigma$ . Let  $e(-)$  be the topological Euler characteristic. Then we have (see [D-ML-P, 4.2])

$$e(X) + t + 2 = 2e(Y) - 2k$$

where  $t$  is the trace of the involution  $\sigma$  on  $H^2(X, \mathbb{C})$  and  $k$  is the number of the isolated fixed points of  $\sigma$ . If  $X$  and  $Y$  are K3 surfaces and  $\sigma = \iota$  is a Nikulin involution, then  $e(X) = e(Y) = 24$  and  $k = 8$ . Therefore we get  $t = 6$ . Since  $\dim H_{\text{tr}}^2(X) = \dim H_{\text{tr}}^2(Y)$  and  $b_2(X) = b_2(Y) = 22$ , we have  $\rho(X) = \rho(Y)$ .  $\square$

**Theorem 27.** *Let  $X$  be K3 surface with a Nikulin involution  $\iota$ . If  $\mathfrak{h}(X)$  is finite dimensional then  $\mathfrak{h}(X) \simeq \mathfrak{h}(Y)$ , therefore  $\iota$  acts as the identity on  $A_0(X)$ .*

*Proof.* Let  $Y$  be the desingularization of  $X/\iota$ . Then  $Y$  is a K3 surface and we have  $\mathfrak{t}_2(\tilde{X}) \simeq \mathfrak{t}_2(X)$  because  $\mathfrak{t}_2(-)$  is a birational invariant for surfaces, see [K-M-P]. Also

$$H_{\text{tr}}^2(X) \simeq H_{\text{tr}}^2(\tilde{X}) \simeq H_{\text{tr}}^2(Y)$$

because the Nikulin involution acts trivially on  $H_{\text{tr}}^2(X)$ . Let  $E_i, 1 \leq i \leq 8$  be the exceptional divisors of the blow-up  $\tilde{X} \rightarrow X$  and let  $g_*(E_i) = C_i$  be the corresponding  $(-2)$ -curves on  $Y$ . We have  $\rho = \text{rank}(\text{NS}(X)) \geq 9$ ,  $b_2(X) = b_2(Y) = 22$  and  $e(X) = e(Y) = 24$ , where  $e(X)$  is the topological Euler characteristic. Let  $t$  be the trace of the action of the involution  $\iota$  on the vector space  $H^2(X, \mathbb{C})$ . By Lemma 26 we have  $t = 6$ . The involution  $\iota$  acts trivially on  $H_{\text{tr}}^2(X)$  which is a subvector space of  $H^2(X, \mathbb{C})$  of dimension  $22 - \rho$ ; therefore the trace of the action of  $\iota$  on  $\text{NS}(X) \otimes \mathbb{C}$  equals  $\rho - 16$ . Since the only eigenvalues of an involution are  $+1$  and  $-1$  we can find an orthogonal basis for  $\text{NS}(X) \otimes \mathbb{C}$  of the form  $H_1, \dots, H_r; D_1, \dots, D_8$ , with  $r = \rho - 8 \geq 1$  such that  $\iota_*(H_j) = H_j$  and  $\iota_*(D_l) = -D_l$ . Then  $\text{NS}(\tilde{X}) \otimes \mathbb{C}$  has a basis of the form  $E_1, \dots, E_8; H_1, \dots, H_r; D_1, \dots, D_8$ . Since  $X$  and  $Y$  are K3 surfaces we have  $q(X) = q(Y) = q(\tilde{X}) = 0$ . Therefore we can find Chow-Künneth decompositions for the motives  $\mathfrak{h}(X), \mathfrak{h}(\tilde{X})$  such that  $\mathfrak{h}_1 = \mathfrak{h}_3 = 0$  and

$$\mathfrak{h}(X) = \mathbb{1} \oplus \mathfrak{h}_2^{\text{alg}}(X) \oplus \mathfrak{t}_2(X) \oplus \mathbb{L}^2 \simeq \mathbb{1} \oplus \mathbb{L}^{\oplus \rho} \oplus \mathfrak{t}_2(X) \oplus \mathbb{L}^2$$

$$\mathfrak{h}(\tilde{X}) = \mathbb{1} \oplus \mathfrak{h}_2^{\text{alg}}(\tilde{X}) \oplus \mathfrak{t}_2(X) \oplus \mathbb{L}^2 \simeq \mathfrak{h}(X) \oplus \mathbb{L}^{\oplus 8}$$

where  $\mathfrak{h}_2^{\text{alg}}(\tilde{X}) = (\tilde{X}, \pi_2^{\text{alg}}(\tilde{X}))$  with  $\pi_2^{\text{alg}}(\tilde{X}) = \Gamma + I$  and

$$\Gamma = \sum_{1 \leq k \leq 8} \frac{[E_k \times E_k]}{E_k^2} + \sum_{1 \leq j \leq r} \frac{[H_j \times H_j]}{H_j^2}, \quad I = \sum_{1 \leq h \leq 8} \frac{[D_h \times D_h]}{D_h^2}.$$

Also

$$\mathbb{L}^{\oplus 8} \simeq \left( \tilde{X}, \sum_{1 \leq k \leq 8} \frac{[E_k \times E_k]}{E_k^2} \right)$$

Let  $g: \tilde{X} \rightarrow Y$  and let  $p = 1/2(\Gamma_g^t \circ \Gamma_g) \in A^2(\tilde{X} \times \tilde{X})$ : then  $p$  is a projector and

$$\mathfrak{h}(\tilde{X}) = (\tilde{X}, p) \oplus (\tilde{X}, \Delta_{\tilde{X}} - p) \simeq \mathfrak{h}(Y) \oplus (\tilde{X}, \Delta_{\tilde{X}} - p)$$

because  $(\tilde{X}, p) \simeq \mathfrak{h}(Y)$  by [Ma, §3 Example 1]. The set of  $r + 8 = \rho$  divisors  $g_*(E_k) = C_k$ , for  $1 \leq k \leq 8$  and  $g_*(H_j) \simeq H_j$ , for  $1 \leq j \leq r$  gives an orthogonal basis for  $\text{NS}(Y) \otimes \mathbb{Q}$ . Therefore we can find a Chow-Künneth decomposition of  $\mathfrak{h}(Y)$  such that

$$\mathfrak{h}_2^{\text{alg}}(Y) \simeq (\tilde{X}, \Gamma) \simeq \mathbb{L}^{\oplus \rho}$$

and we get

$$\mathfrak{h}_2(\tilde{X}) = \mathfrak{h}_2^{\text{alg}}(\tilde{X}) \oplus \mathfrak{t}_2(X) \simeq \mathfrak{h}_2^{\text{alg}}(Y) \oplus \mathbb{L}^{\oplus 8} \oplus \mathfrak{t}_2(Y) \oplus M$$

where  $H^*(M) = 0$  because  $H_{\text{tr}}^2(\tilde{X}) = H_{\text{tr}}^2(X) = H_{\text{tr}}^2(Y)$ . From Theorem 6 it follows that  $M = 0$  and we get an isomorphism

$$\mathfrak{h}_2(\tilde{X}) \simeq \mathfrak{h}_2(Y) \oplus \mathbb{L}^{\oplus 8} \simeq \mathfrak{h}_2(X) \oplus \mathbb{L}^{\oplus 8}$$

which implies  $\mathfrak{h}(X) \simeq \mathfrak{h}(Y)$ . The rest follows from Proposition 25.  $\square$

The following result gives examples of K3 surfaces with a Nikulin involution  $\iota$  such that  $\iota$  acts as the identity on  $A_0(X)$ .

**Theorem 28.** *Let  $X$  be a smooth projective K3 surface over  $\mathbb{C}$  with  $\rho(X) = 19, 20$ . Then  $X$  has a Nikulin involution  $\iota$ ,  $\mathfrak{h}(X)$  is finite dimensional and  $\iota$  acts as the identity on  $A_0(X)$ .*

*Proof.* By [Mo, 6.4]  $X$  admits a Shioda-Inose structure, i.e. there is a Nikulin involution  $\iota$  on  $X$  such that the desingularization  $Y$  of the quotient surface  $X/\iota$  is a Kummer surface, associated to an abelian surface  $A$ ; hence  $\mathfrak{h}(Y)$  is finite dimensional by [Pe1, 5.8]. The rational map  $f: X \rightarrow Y$  induces a splitting  $\mathfrak{t}_2(X) \simeq \mathfrak{t}_2(Y) \oplus N$ . Since  $\mathfrak{t}_2(Y)$  is finite dimensional we are left to show that  $N = 0$ . By the same argument as in the proof of Proposition 25 the vanishing of  $N$  is equivalent to  $A_0(X)^\iota = A_0(Y)$ . By [Mo, 6.3 (iv)] the Neron Severi group of  $X$  contains the sublattice  $E_8(-1)^2$ . Hence by the results in [Hu2, 6.3, 6.4] the symplectic automorphism  $\iota$  acts as the identity on  $A_0(X)$ . As, by [K-M-P, 6.13], we have  $\mathfrak{t}_2(Y) = \mathfrak{t}_2(A)$ , the motive  $\mathfrak{h}(X)$  is finite dimensional and it lies in the subcategory of  $\mathcal{M}_{\text{rat}}(\mathbb{C})$  generated by the motives of abelian varieties.  $\square$

The next theorem gives examples of surfaces  $X$  and  $Y$  such that  $M(X) \simeq M(Y)$  but the derived categories  $D^b(X)$  and  $D^b(Y)$  are not equivalent. We will use the following result by Van Geemen and Sarti

**Proposition 29.** ([VG-S 2.5]) *Let  $X$  be a complex K3 surface with a Nikulin involution  $\iota$  and let  $Y$  be a desingularization of the quotient surface  $X/\iota$ . The involution induces an isomorphism of Hodge structures between  $T_{X,\mathbb{Q}}$  and  $T_{Y,\mathbb{Q}}$ . If the dimension of the  $\mathbb{Q}$ -vector space  $T_{X,\mathbb{Q}}$  is odd there is no isometry between  $T_{X,\mathbb{Q}}$  and  $T_{Y,\mathbb{Q}}$ .*

**Theorem 30.** *Let  $X$  be a complex K3 surface with a Nikulin involution  $\iota$  such that  $\rho(X) = 9$  and let  $Y$  be the desingularization of  $X/\iota$ . Assume that the map  $f: X \rightarrow Y$  induces an isomorphism between  $\mathbf{t}_2(X)$  and  $\mathbf{t}_2(Y)$ . Then  $\iota$  acts as the identity on  $A_0(X)$ , the rational map  $f: X \rightarrow Y$  induces an isomorphism  $M(X) \xrightarrow{\sim} M(Y)$  in  $DM_{gm}(\mathbb{C})$ , but the isomorphism of Hodge structures  $\phi_\iota: T_{X,\mathbb{Q}} \rightarrow T_{Y,\mathbb{Q}}$  is not an isometry.*

*Proof.* The Nikulin involution  $\iota$  induces an isomorphism of Hodge structures  $\phi_\iota: T_{X,\mathbb{Q}} \rightarrow T_{Y,\mathbb{Q}}$  which by Proposition 29 is not an isometry because  $\dim T_{X,\mathbb{Q}} = 22 - 9$  is odd. Since  $X$  and  $Y$  are both K3 surfaces the isomorphism  $\mathbf{t}_2(X) \simeq \mathbf{t}_2(Y)$  implies  $\mathfrak{h}(X) \simeq \mathfrak{h}(Y)$  in  $\mathcal{M}_{\text{rat}}^{\text{eff}}(\mathbb{C})$ , hence also  $M(X) \simeq M(Y)$ .  $\square$

**Examples 31.** The following are examples of K3 surfaces  $X$  with a Nikulin involution  $\iota$  and  $\rho(X) = 9$  such that  $\mathbf{t}_2(X) \simeq \mathbf{t}_2(Y)$  hence  $\mathfrak{h}(X) \simeq \mathfrak{h}(Y)$ . Therefore  $X$  satisfies Huybrechts' conjecture 3, i.e.  $\iota$  acts as the identity on  $A_0(X)$ . On the other hand,  $X$  and  $Y$  are not Fourier-Mukai partner because, as in Theorem 30, there is no Hodge isometry between their transcendental lattices. The proof of the isomorphism  $\mathbf{t}_2(X) \simeq \mathbf{t}_2(Y)$  in these cases follows directly from the geometric description of  $X$  and  $Y$  given by Van Geemen and Sarti in [VG-S], see [Pe2].

(i)  $X$  a double cover of  $\mathbb{P}^2$  branched over a sextic curve and  $Y$  a double cover of a quadric cone in  $\mathbb{P}^3$ ;

(ii)  $X$  is a double cover of a quadric in  $\mathbb{P}^3$  and  $Y$  is the double cover of  $\mathbb{P}^2$  branched over a reducible sextic;

(iii)  $X$  is the intersection of 3 quadrics in  $\mathbb{P}^5$  and  $Y$  is a quartic surface in  $\mathbb{P}^3$ .

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